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Entropy of the Rindler Horizon

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The entropy of the Rindler horizon to a nonuniformly accelerating observer is investigated. It is shown that result proportional to the area relies on a time-dependent cutoff, and the local energy determined by the cutoff is closer to the Planck scale than the brick-wall model. The method and result obtained in this paper can well be applied to the nonstationary black hole.

The line element (Misner *et al.*, 1973)

$$ds^2 = -(1 + ax)^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

describes a uniformly accelerating Rindler observer in Minkowski space–time, where the acceleration a is a constant. Although an inertial observer is in the Minkowski vacuum state, it is shown that the Rindler observer is in a thermal bath and receives thermal radiation from the horizon located by $\xi = -1/a$. It is interesting that an equation similar to (1) is still a solution of Einstein's equation when a is time-dependent (Tang, 1989), $a = a(t)$. According to our understanding of it, the line element (Tang, 1989)

$$ds^2 = -[1 + a(t)x]^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

describes an observer with a variable acceleration. The horizon is located by (Zhao and Luo, 1992)

$$\xi = -\frac{1}{a}(1 - \dot{\xi}), \tag{3}$$

where $\dot{\xi} = d\xi/dt$. The Hawking–Unruh effect has also been investigated and it has been shown that the temperature is proportional to the variable acceleration (Zhao and Luo, 1992)

$$T(t) = \frac{a(t)}{2\pi}. \tag{4}$$

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This shows that on a large scale the Rindler space with an inconstant acceleration is not in thermodynamic equilibrium because the temperature is time-dependent. However, as a two-dimensional system, the horizon is in equilibrium because of the plane symmetry. The method to study the Hawking–Unruh radiation of the Rindler horizon with inconstant acceleration is based on the Damour–Ruffini scheme (Damour and Ruffini, 1976). The essential point of this scheme is that the equations of the quantum fields near the horizon asymptotically approach the form

$$\frac{\partial^2\Phi}{\partial r_*^2} - \frac{\partial^2\Phi}{\partial t^2} = 0, \tag{5}$$

or other equivalent form [see Eq. (7)], where r_* is the tortoise coordinate. The in-coming and out-going waves can be obtained. The out-going solution is not analytical at the horizon. However, it can be extended to the interior of the horizon. Following Damour and Ruffini, the spectral distribution is given by the interior production of the wave functions.

In the Rindler space–time with inconstant acceleration, the generalized tortoise coordinates are defined as

$$\begin{aligned} v &= t - t_0, dv = dt, \\ r_* &= x + \frac{1}{2\kappa}\ln[x - \xi(t)], \end{aligned} \tag{6}$$

where $\kappa = \kappa(t_0)$ is a parameter which is treated as a constant under the above coordinate transformations. We demand that the equation of quantum field near the horizon be asymptotically deduced to the following equation

$$\frac{\partial^2\Phi}{\partial r_*^2} + 2\frac{\partial^2\Phi}{\partial v\,\partial r_*} = 0. \tag{7}$$

Thus, both κ and ξ are determined: $\kappa = a(t)$, ξ is just the location of the horizon shown by Eq. (3), which is consistent with the null condition. By using Damour and Ruffini scheme, it is shown that the parameter κ appears in the spectrum and is proportional to the radiation temperature, $T = a(t)/2\pi$.

The similar method can extensively and successfully be applied to the non-static black holes. The relevant works are in Zhao and Dai (1992), Yang and Zhao (1993), and Li *et al*. (1998).

It is interesting to compute the entropy of the horizon in the nonuniformly accelerating Rindler space–time. The following is devoted to this problem.

Comparing Eq. (2) with (3), one can see that the infinite red-shift surface does not coincide with the horizon. We expect that there exists a frame where the two surfaces are identical. We introduce the following coordinate transformation

$$x_* = x - \xi,\; dx_* = dx - \dot{\xi}\,dt, \tag{8}$$

then Eq. (2) can be reduced to

$$ds^2 = -[(1 + ax)^2 - \dot{\xi}^2]\,dt^2 + dx_*^2 + 2\dot{\xi}\,dtdx_* + dy^2 + dz^2. \qquad (9)$$

The physical meaning of the coordinate transformation is easily understood. In order to cancel the effect caused by the variability of the horizon, we must choose a frame co-moving with the horizon. In Rindler system, an observer co-moving with the event horizon is described by $dx_* = 0$. A surface just outside the horizon is fixed at $x_* = \epsilon,\ \epsilon$ is a small quantity and $d\epsilon = 0$ is required. Thus the geometry of this surface is determined by

$$ds_3^2 = -[(1 + ax)^2 - \dot{\xi}^2]dt^2 + dy^2 + dz^2, \qquad (10)$$

and

$$\sqrt{-g} = \sqrt{-g_{00}} = \sqrt{(1 + ax)^2 - \dot{\xi}^2},\ g^{00} = \frac{1}{g_{00}},\ g^{11} = g^{22} = 1. \qquad (11)$$

Substituting (11) into the following equation of massless scalar field

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = 0, \qquad (12)$$

we obtain

$$-\frac{g'_{00}}{2(-g_{00})^{3/2}}\partial_t\Phi - \frac{1}{\sqrt{-g_{00}}}\partial_t^2\Phi + \sqrt{-g_{00}}\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\Phi = 0, \qquad (13)$$

where $g'_{00} = \partial_t g_{00}$. The first term will be vanishing in the case of the uniform acceleration, $g'_{00} = 0$. The corresponding solution reads

$$\Phi = f(\epsilon)\,e^{-i\omega t + iS(\epsilon) + i(yk_y + zk_z)}. \qquad (14)$$

In the case of inconstant acceleration, generally, the solution can be supposed as

$$\Phi = F(t, \epsilon)\,e^{i(yk_y + zk_z)}, \qquad (15)$$

ϵ is the small parameter, $d\epsilon = 0$, as mentioned in the previous part. It is useful to know the asymptotic behavior of equation near the horizon because we only investigate the field in the vicinity of the horizon. It is shown in Zhao and Luo (1992)

$$F(t, \epsilon) \sim e^{-i\omega t}, \qquad (16)$$

as $\epsilon \to 0$, which is a trivial solution of Eq. (7). It means that Eq. (14) can be treated as the zeroth approximation of solution of Eq. (12). In general, if we suppose

$$\Phi \sim f(t, \epsilon)\,e^{-i\omega t + iS(\epsilon) + i(yk_y + zk_z)}, \qquad (17)$$

in the vicinity of the horizon. Substituting (17) into (13), we have

$$-\frac{g_{\dot{0}0}}{2}\partial_t f + g_{00}\left[\partial_t^2 f - \omega^2 f\right] - g_{00}^2\left(k_y^2 + k_z^2\right) f = 0. \qquad (18)$$

We notice

$$\partial_t f \to 0, \tag{19}$$

as $g_{00} \to 0$. This means f asymptotically approaches a constant. It is an evidence for Eq. (16). Substituting (14) into (13), we obtain

$$k^2 = k_y^2 + k_z^2 = \frac{\omega^2}{-g_{00}} = E^2. \tag{20}$$

It is the momentum–energy relation for an instant t. $E = \omega/\sqrt{-g_{00}}$ is the locally defined energy. For a finite part of the Rindler horizon with area A, the number of quantum states in the momentum range $(k, k + dk)$ reads

$$dN(k) = \frac{2\pi Akdk}{(2\pi)^2}, \tag{21}$$

or in the energy range $(E, E + dE)$

$$dN(E) = \frac{AEdE}{2\pi}. \tag{22}$$

The logarithm of the partition function is defined as

$$\begin{aligned}\ln Z &= -\int dN(E)\ln(1 - e^{-\beta E}) = \frac{A}{4\pi}\int \frac{E^2 d(\beta E)}{e^{\beta E} - 1}\\ &= \frac{A}{4\pi\beta^2}\int_0^\infty \frac{p^2 dp}{e^p - 1} = \frac{\zeta(3)A}{2\pi\beta^2},\end{aligned} \tag{23}$$

where $\beta = T^{-1}\sqrt{-g_{00}}$, $p = \beta E$, and the zeta function $\zeta(3) = 1.202$. The entropy reads

$$S = \ln Z - \beta\frac{\partial \ln Z}{\partial\beta} = \frac{3\zeta(3)A}{2\pi\beta^2} = -\frac{3\zeta(3)A}{2\pi g_{00}}T^2, \tag{24}$$

where $g_{00}(\epsilon) \sim \epsilon$. Obviously, entropy becomes infinite when ϵ approaches zero. However, the nonzero cutoff is necessary if the locally defined energy E of mode is not allowed to exceed the Planck scale. According to Wien's displacement law, the maximum of energy density of black body radiation is at the specific mode with frequency $\omega_{\max} = 2.822T$. The local energy near the horizon is given by (in the static case)

$$E_{\max} = \frac{\omega_{\max}}{\sqrt{-g_{00}}} = \frac{2.822}{\sqrt{-g_{00}}}T. \tag{25}$$

Therefore, the square $E_{\max}$ appears in (24)

$$S = \frac{3\zeta(3)E_{\max}^2}{2\pi \times (2.822)^2}A. \tag{26}$$

The fine computation shows that entropy will be reduced to the standard Bekenstein–Hawking formula, if $E_{\text{max}} = 1.86$ (Planck energy).

The nonzero cutoff is reasonable. According to the general concepts of the quantum field theory in curved space–time, there exists an observable minimal length: the length less than Planck length is not observable. In other words, the modes with energy higher than Planck energy is nonobjective. In our understanding, those modes with energy higher than Planck scale have no contribution to the entropy observed by a Rindler observer. There exists a similar case, Pauli–Villars regularization of quantum field theory (Demers *et al.*, 1995). In that scheme, An assistant field with infinite mass is introduced. However, according to uncertainty principle, the field is nonobjective and doesn't produce any physical effect.

The reason why we investigate the entropy of the Rindler horizon is stimulated by the geometric character of black hole entropy. We recall the equation of the state of thermal radiation in flat space–time, where entropy is an extensive quantity and is proportional to the volume. However, this is only valid for three-dimensional system. The entropy will be proportional to the area if a two-dimensional system is investigated. Even in brick-wall model ('t Hooft, 1985) and entanglement interpretation (Bombelli *et al.*, 1986; Frolov and Novikov, 1993; Srednicki, 1993), it is shown that the main contribution to the entropy of the black hole is attributed to the modes in close vicinity of the horizon. Therefore, the geometry of near horizon region merits consideration. It is well known that the geometry closely near the horizon of a static hole is Rindler-like

$$ds^2 = -X^2 dt^2 + dX^2 + r_0^2 d\Omega^2. \tag{27}$$

We introduce the following transformations

$$X = \frac{1}{\kappa} + x,\ t = \kappa t', \tag{28}$$

Eqs. (27) becomes

$$ds^2 = -(1+\kappa x)^2 dt'^2 + dx^2 + r_0^2 d\Omega^2, \tag{29}$$

where κ is the surface gravity at the static horizon. In the infinitesimal vicinity of a point at the horizon, the metric of plane can substitute for the geometry of spherical surface. Then, Eqs. (1), point by point, describes the geometry of an infinitesimal vicinity to the horizon. The entropy of the black hole can be obtained by Rindler approximation (Frolov and Fursaev, 1998; Li and Zhao, 2000a).

Most authors focus their attention on the stationary black holes. However, considering the Hawking radiation of holes, the more real space–time is nonstationary. It is meaningful and interesting to compute the entropy of nonstationary black hole. By using the condition of linear nonequilibrium, we have investigated the entropy of Vaidya black hole (Li and Zhao, 2000b). We believe that the result obtained in this paper can also be applied to the nonstationary case. Although the surface gravity is not well-defined in a nonstationary space–time, we still know by intuition that an observer near the horizon feels a variable acceleration. This is very similar to the case of nonuniformly accelerating observer in Minkowski space–time.

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